Perturbation theory for solitons of the Manakov system

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We formulate the Kaup-Karpman-Maslov-type perturbation theory for solitons connected with the 3×3 matrix spectral problems, using the example of the perturbed Manakov system. The adiabatic approximation and first-order corrections to the soliton shape are considered. The self-orientation effect of the soliton polarization dynamics caused by a cubic perturbation is described. It is also shown that the combined action of linear and cubic perturbations provides existence of a stationary regime for the soliton propagation with the single fixed amplitude and that the corrections to the soliton shape from linear and cubic perturbations partially compensate each other. [S1063-651X(97)16405-4]

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I. INTRODUCTION

At the present time much attention is being paid to the investigation of optical soliton polarization dynamics. The interest in the polarization effects is primarily caused by the necessity to take into account birefringence of real optical fibers [1–9]. Single-mode optical fibers support two modes of polarization due to linear birefringence combined with weak intermodal dispersion. These modes are coupled together by means of the Kerr effect which stabilizes solitons against spreading due to dispersion and against broadening and splitting due to birefringence. Secondarily, polarization dynamics is closely related to the cross-phase modulation which, in many cases, leads to formation of the bound states of solitons [4,5]. This effect has also attracted attention in a number of applications connected with pulse compression [10], short-pulse generation [11], and wavelength-division multiplexing [12]. Finally, we point out the connection of the studied issue to the recent studies of optical domain walls separating the regions of different polarizations [13–15].

Taking into account the polarization of an electromagnetic field, the propagation of light pulses in a Kerr medium is described by a system of coupled nonlinear Schrödinger equations (CNLS) [8]. In general, such a system is nonintegrable in terms of the inverse scattering transform (IST) method. Numerical simulations [2,3,16–18] have revealed a much richer dynamics for CNLS, as compared with a single NLS equation, although, they do not provide as much universality as analytical methods.

There are different approaches to the analytical description of polarized (vector) soliton dynamics. One of the frequently used methods deals with various *Ansätze* which reduce CNLS equations to a system of ordinary differential equations [5,14,17–20]. The variational approach was successfully applied in Refs. [21,22]. It permits us, in particular, to obtain the conditions of resonance splitting of a two-component soliton into two separating solitons of different polarizations. By means of the Lie group analysis, Alfinito *et al.* [23] found some exact solutions of CNLS with reference to the linear birefringence. At last, in Refs. [24,25] some elements of perturbation theory for polarized solitons were developed. The idea of this approach is based on the fact that for some values of parameters the original CLNS system transforms to the Manakov system [26]

$$
\begin{aligned}
&iq_{1z}+\tfrac{1}{2}q_{1\tau\tau}+(|q_1|^2+|q_2|^2)q_1=0,\\
&iq_{2z}+\tfrac{1}{2}q_{2\tau\tau}+(|q_2|^2+|q_1|^2)q_2=0,
\end{aligned}
\tag{1.1}
$$

which is integrable by IST method. Here q_1 and q_2 are normalized envelopes of the two modes of polarization, τ and z are, correspondingly, normalized time and distance along the fiber. Under condition that terms violating the integrability of Eq. (1.1) are small, one can estimate their effect on the initially unperturbed vector soliton in the framework of perturbation theory. It should be stressed that the Manakov system is a good approximation to the real physical models. For example, Kaup and Malomed [27] have proved that such phenomena as soliton trapping and daughter wave (shadow) formation encountered in optical fibers are already contained in the Manakov model. Moreover, in a recent paper [28] the Manakov system was used to study the polarization scattering of soliton-soliton collisions.

Hence, from the point of view of an analytical approach to investigation of polarized solitons, the use of the Manakov system as a zero-order approximation, followed by the account of small nonintegrable corrections, is very promising. Such an approach in the case of nonlinear equations integrable by the 2×2 matrix version of the IST method (in particular, the famous NLS equation) is based on the well-known scheme by Kaup-Karpman-Maslov (KKM) [29,30]. At the same time, the formalism of perturbation theory for soliton equations integrable by the 3×3 matrix version of the IST method [e.g., the Manakov system (1.1)], which would be of the same completeness and convenience in calculation as the KKM scheme, has not been constructed. The reason for that is mainly the mathematical peculiarities of the spectral problems over the space of 2×2 matrices. Indeed, one finds that the matrix Jost solutions to the spectral problems of 2×2 matrix dimension possess definite analytic properties with respect to a spectral variable as a whole, thus allowing us to use the methods of analysis in the complex plane of the spectral variable directly. On the other hand, being turned to higher matrix dimensions, one faces the problem of indefinite analytic behavior of the matrix Jost solutions. Hence, some preliminary manipulations with the Jost solutions have to be carried out to adapt the KKM approach to higher matrix dimensions.

In the present paper we propose a simple formalism of perturbation theory to soliton equations associated with the 3×3 matrix spectral problems, which is as efficient as the KKM method. Since the key element of the proposed formalism is analytic behavior of the Jost-type functions, the Riemann-Hilbert (RH) problem (see, for example, Ref. [31]) is the natural basis for our method. The first application of the RH problem to perturbed nonlinear equations associated with the 2×2 matrix spectral problem was made by Kivshar [32] for calculation of the first-order corrections to the soliton of the Landau-Lifshitz equation. In Refs. [33] the possibility of a purely algebraic calculation of higher-order corrections to the NLS soliton on the basis of the RH problem was pointed out and dynamics of a perturbed optical soliton in a fiber with combined resonant and nonresonant (cubic) nonlinearities was described.

In Sec. II we summarize the results of the RH-based approach to the unperturbed Manakov system. In particular, we give a simple derivation of the soliton solution. The perturbation-induced evolution equations for discrete spectral data are obtained in Sec III. On the basis of these equations in Sec. IV, for the case of combined action of linear and cubic perturbations to the Manakov system, we calculate adiabatic corrections to the soliton parameters. We also prove asymptotic stabilization of the polarization modes which is dependent on the initial polarization state. Moreover, we find the stationary regime of perturbed soliton propagation and the value of a steady-state soliton amplitude. In Sec. V we calculate the first-order corrections to the soliton shape and give evidence of partial compensation in the stationary regime of corrections caused by linear and cubic perturbations. In concluding Sec. VI we point out some ways to generalize proposed soliton perturbation theory. The Appendix is devoted to detailed derivation of the equations for the perturbation-induced evolution of the discrete spectral data which are used in Sec. IV.

II. THE RIEMANN-HILBERT PROBLEM AND SOLITON OF THE MANAKOV SYSTEM

The Manakov system (1.1) is integrable by means of the IST method [26,31] and can be represented as a compatibility condition for the following system of two linear matrix equations:

$$\Psi_\tau = U\Psi - ik\Psi A, \quad \Psi_z = V\Psi - ik^2\Psi A, \tag{2.1}$$

where

$$U = i(kA+Q), \quad V = ik^2A + ikQ + \tfrac{1}{2}AQ_\tau - \frac{i}{2}AQ^2,$$

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \quad Q=\begin{pmatrix}0&0&q_1^*\\0&0&q_2^*\\q_1&q_2&0\end{pmatrix}. \tag{2.2}$$

In other words, Eqs. (1.1) are equivalent to $U_\tau - V_z + [U,V]=0 \Leftrightarrow iQ_z - \frac{1}{2}AQ_{\tau\tau} - AQ^3 = 0$ for all values of the spectral variable k. The following analysis is based on the RH problem associated with the spectral equation:

$$\Psi_\tau = ik[A,\Psi] + iQ\Psi. \tag{2.3}$$

In this connection we demonstrate below the way to obtain the soliton solution of the Manakov system within the RH framework and define the notations used.

Consider the spectral problem (2.3) with the potential Q defined in Eq. (2.2). We assume the functions $q_j(\tau,z)$, $j=1,2$, belonging to the Schwarz space ($q_j\to0$ as $|\tau|\to\infty$). Define the matrix Jost solutions $\Psi_\pm$ of Eq. (2.3) which satisfy the asymptotic conditions $\Psi_\pm\to\mathbb{1}$ for $\tau\to\pm\infty$, $\mathbb{1}$ is the unit matrix. The scattering matrix $S(k)$ can be defined in terms of the Jost solutions

$$\Psi_-(k,\tau) = \Psi_+(k,\tau)ES(k)E^{-1}, \quad E\equiv\exp(ikA\tau).$$

The RH problem associated with the spectral equation (2.3) can be derived in the following way [31]. Perform a factorization of the scattering matrix S,

$$S_+ = SS_-, \tag{2.4}$$

providing the entries of the matrices $S_\pm$ are expressed without division in terms of the entries of the matrix S. In Ref. [26] it was shown that some rows and columns of the matrices $\Psi_\pm$ possess analytic properties with respect to k. In particular, the columns $(\Psi_+)_{\cdot1}$, $(\Psi_+)_{\cdot2}$, and $(\Psi_-)_{\cdot3}$ are holomorphic for $\mathrm{Im}k>0$, while the rows $(\Psi_+^{-1})_{1\cdot}$, $(\Psi_+^{-1})_{2\cdot}$, and $(\Psi_-^{-1})_{3\cdot}$ are holomorphic for $\mathrm{Im}k<0$. This enables us to fix the matrix S_+. We assume

$$S_+=\begin{pmatrix}1&0&S_{13}\\0&1&S_{23}\\0&0&S_{33}\end{pmatrix}, \tag{2.5}$$

where S_{ij} is an entry of the scattering matrix S. Then the following matrix [33]:

$$\Phi_+ = \Psi_+ES_+E^{-1} \equiv [(\Psi_+)_{\cdot1},(\Psi_+)_{\cdot2},(\Psi_-)_{\cdot3}] \tag{2.6}$$

is holomorphic in the upper half-plane of the complex k plane, while the matrix (the superscript t stands for the transpose)

$$\Phi_-^{-1} = ES_+^\dagger E^{-1}\Psi_+^{-1} \equiv ((\Psi_+^{-1})_{1\cdot},(\Psi_+^{-1})_{2\cdot},(\Psi_-^{-1})_{3\cdot})^t \tag{2.7}$$

is holomorphic in the lower half-plane. These two matrices represent the solution of the RH problem

$$\Phi_-^{-1}\Phi_+ = ES_+^\dagger S_+E^{-1} \equiv G, \quad \Phi_\pm\to\mathbb{1}, k\to\infty, \tag{2.8}$$

on the analytic factorization of the nondegenerate matrix $G(k)$ given on the real axis $\mathrm{Im}k=0$. In addition, the matrices $\Phi_\pm$ satisfy the spectral equation (2.3). Substituting the asymptotic decomposition in k,

$$\Phi_+=\mathbb{1}+k^{-1}\Phi_+^{[1]}+\cdots \tag{2.9}$$

into Eq. (2.3) one reconstructs the potential Q from the solution of the RH problem (2.8)

$$Q=-[A,\Phi_+^{[1]}]. \tag{2.10}$$

In the general case the matrices Φ_+ and Φ_- have zeros k_j and $\bar{k}_j$, $j=1,\ldots,N$, in the upper and lower half-planes, respectively, i.e., $\det\Phi_+(k_j)=0$ and $\det\Phi_-^{-1}(\bar{k}_j)=0$. Thus we have in general the RH problem with zeros. By virtue of the Hermiticity of the potential Q, the matrix $G(k)$ is also Hermitian one and the matrices $\Phi_\pm$ satisfy the involution condition $\Phi_+^\dagger(k)=\Phi_-^{-1}(k^*)$. It follows that the index of the RH problem (2.8) [or, more exactly, the index of the $\det G(k)$ over the real axis $\mathrm{Im}k=0$] is zero, and as a result we have paired zeros, i.e., every zero k_j in the upper half-plane has its counterpart $\bar{k}_j=k_j^*$ in the lower half-plane. The solution $\Phi_\pm$ of the RH problem with zeros can be expressed through the solution $\Phi_\pm^0$ of the regular RH problem,

$$\Phi_\pm=\Phi_\pm^0\Gamma,\quad \det\Phi_\pm^0\neq 0\quad \forall k,\quad \mathrm{Im}k\neq 0, \tag{2.11}$$

with

$$(\Phi_-^0)^{-1}\Phi_+^0=\Gamma G\Gamma^{-1}\equiv G^0. \tag{2.12}$$

Soliton solutions of the Manakov system (1.1) correspond to the RH problem with zeros provided $G=\mathbb{1}$, i.e., $\Phi_\pm^0=\mathbb{1}$, the matrix Γ being expressed through some projective matrices. For example, one-soliton solution corresponds to a single zero k_1 in the upper half-plane and the matrix Γ reads [31]

$$\Gamma(k,\tau,z)=\mathbb{1}-\frac{k_1-k_1^*}{k-k_1^*}P(\tau,z), \tag{2.13}$$

where $P(\tau,z)$ is a projective matrix, $P=|p\rangle(\langle p|p\rangle)^{-1}\langle p|$,

$$P=(|p_1|^2+|p_2|^2+|p_3|^2)^{-1}\begin{pmatrix}|p_1|^2 & p_1p_2^* & p_1p_3^*\\ p_2p_1^* & |p_2|^2 & p_2p_3^*\\ p_3p_1^* & p_3p_2^* & |p_3|^2\end{pmatrix},$$

composed out of the entries of the three-component vector column $|p\rangle=(p_1,p_2,p_3)^t$. The vector $|p\rangle$ is a solution, up to an arbitrary norm, of the equation $\Gamma(k_1)|p\rangle=0$. Comparing Eqs. (2.9) and (2.13) we conclude that $\Phi_+^{[1]}=-(k_1-k_1^*)P$, which allows us to express the potential through the projective matrix P: $Q=(k_1-k_1^*)[A,P]$ or

$$q_j=-2(k_1-k_1^*)P_{3j},\quad j=1,2. \tag{2.14}$$

Therefore, the problem of finding the soliton solution is reduced to finding the projective matrix P. Its dependence on the coordinates τ and z is determined by the equations

$$|p\rangle_\tau=ik_1A|p\rangle,\quad |p\rangle_z=ik_1^2A|p\rangle, \tag{2.15}$$

hence it follows that ($k_1=\xi+i\eta$), $j=1,2$,

$$p_j(\tau,z)=\exp[i\xi\tau+i(\xi^2-\eta^2)z]\exp(-\eta\tau-2\xi\eta z)p_j^0,$$

$$p_3(\tau,z)=\exp[-i\xi\tau-i(\xi^2-\eta^2)z]\exp(\eta\tau+2\xi\eta z)p_3^0$$

where p_j^0 and p_3^0 are constants determined by the initial conditions imposed on Eqs. (2.15). By virtue of the fact that the matrix P is independent of the norm of $|p\rangle$, we can express P in terms of the quantities n_j, $n_j=(p_j/p_3)^*$,

$$P=(|n_1|^2+|n_2|^2+1)^{-1}\begin{pmatrix}|n_1|^2 & n_1^*n_2 & n_1^*\\ n_1n_2^* & |n_2|^2 & n_2^*\\ n_1 & n_2 & 1\end{pmatrix}, \tag{2.16}$$

where $n_j^0=(p_j^0/p_3^0)^*$,

$$n_j(\tau,z)=\exp[-2i\xi\tau-2i(\xi^2-\eta^2)z]\times\exp(-2\eta\tau-4\xi\eta z)n_j^0. \tag{2.17}$$

As the solution $q(\tau,z)$ is given in terms of P_{3j}, $j=1,2$, we note the explicit expression for P_{3j}

$$P_{3j}=\frac{\exp[-2i\xi\tau-2i(\xi^2-\eta^2)z]\exp(-2\eta\tau-4\xi\eta z)}{\exp(-4\eta\tau-8\xi\eta z)(|n_1^0|^2+|n_2^0|^2)+1}n_j^0.$$

Finally, introducing the following notations:

$$|n_1^0|^2+|n_2^0|^2=e^{2\alpha},\quad y=2\eta\tau+4\xi\eta z-\alpha,$$

$$\phi=2\xi\tau+2(\xi^2-\eta^2)z=\frac{\xi}{\eta}y-\Delta(z),$$

$$\Delta(z)=2(\xi^2+\eta^2)z-\frac{\xi}{\eta}\alpha, \tag{2.18}$$

we obtain with the help of Eq. (2.14) the soliton solution of the Manakov system (1.1) [26]

$$q_j(\tau,z)=-2i\eta\Theta_je^{-i\phi}\mathrm{sech}y, \tag{2.19}$$

where $\Theta_j=n_j^0e^{-\alpha}$, $j=1,2$, are the polarization parameters satisfying the natural identity $|\Theta_1|^2+|\Theta_2|^2=1$.

Therefore, given discrete spectral data k_1 and $|p\rangle$ of the RH problem one obtains the one-soliton solution of the Manakov system. Generalization to the case of N zeros k_j, $j=1,\ldots,N$, is straightforward. In the general case, i.e., with N zeros and nonsoliton (continuous) part of spectral data, the latter consist of $G(k,\tau,z)$ for $\mathrm{Im}k=0$ (continuous spectrum) and $k_1,\ldots,k_N$, $|p_1\rangle,\ldots,|p_N\rangle$ with the condition $\mathrm{Im}k_j>0$ (discrete spectrum). A perturbation of system (1.1) will modify z-evolution equations (2.15) of spectral data.

III. PERTURBATION-INDUCED EVOLUTION OF DISCRETE SPECTRAL DATA

The goal of this section is to obtain evolution equations for the discrete spectral data of the RH problem taking into account perturbation. According to the scheme described in Sec. II, this will allow us to find corrections to soliton characteristics.

Recall the spectral equation (2.3). First of all we find variations $\delta\Psi_\pm$ of the functions $\Psi_\pm$ resulting from the variation of potential caused by perturbation R: $\delta Q = R\,\delta z$. From Eq. (2.3) we obtain the equation $(\Psi^{-1}\delta\Psi)_\tau = ik[A,\psi^{-1}\delta\Psi] + i\Psi^{-1}\delta Q\Psi$. Integration with account of $\lim_{\tau\to\pm\infty}(\Psi_\pm^{-1}\delta\Psi)=0$ gives

$$\delta\Psi_\pm = i\Psi_\pm E\left(\int_{\pm\infty}^{\tau} d\tau\, E^{-1}\Psi_\pm^{-1}\delta Q\Psi_\pm E\right)E^{-1}. \quad (3.1)$$

Then corresponding variation of the scattering matrix reads

$$\delta S = \delta(E^{-1}\Psi_+^{-1}\Psi_- E) = iS\int_{-\infty}^{\tau} d\tau\, E^{-1}\Psi_+^{-1}\delta Q\Psi_+ E + i\left(\int_{\tau}^{\infty} d\tau\, E^{-1}\Psi_+^{-1}\delta Q\Psi_+ E\right)S,$$

which together with the factorization (2.4) leads to

$$S_+^{-1}(\delta S)S_- = S_+^{-1}\delta S_+ - S_-^{-1}\delta S_- = i\int_{-\infty}^{\infty} d\tau\, E^{-1}\Phi_+^{-1}\delta Q\Phi_+ E. \quad (3.2)$$

Here the function Φ_+ is defined as in Eq. (2.6) and also satisfies $\Phi_+ = \Psi_+ E S_+ E^{-1} = \Psi_- E S_- E^{-1}$. In the following we shall use a special notation for the integral in the right-hand side of Eq. (3.2), namely,

$$\gamma(a,b) = i\int_a^b d\tau\, E^{-1}\Phi_+^{-1}R\Phi_+ E, \quad (3.3)$$

with the specification $\gamma(-\infty,\infty)\equiv\gamma(k)$. Notice that $\mathrm{tr}\,\gamma(a,b)=0$. We obtain

$$\delta S = S_+\gamma(k)S_-^{-1}\delta z. \quad (3.4)$$

In view of the explicit expression for S_+ (2.5), this variation, as is seen from Eq. (3.4), can be calculated in the following way $[\mathsf{M}_{(33)}\equiv \mathrm{diag}(0,0,1)]$:

$$\delta S = \delta S\mathsf{M}_{(33)} = S_+\gamma(k)S_-^{-1}\mathsf{M}_{(33)}\delta z = S_+\gamma(k)S_+^{-1}S\mathsf{M}_{(33)}\delta z = S_+\gamma(k)\mathsf{M}_{(33)}\delta z. \quad (3.5)$$

Now we can write the variation of the solution to the RH problem. Taking into account from Eq. (3.1) that $\delta\Psi_+ = \Psi_+ E S_+\gamma(\infty,\tau)S_+^{-1}E^{-1}\delta z$, we obtain with the help of Eqs. (2.6) and (3.5)

$$\delta\Phi_+ = \delta\Psi_+ E S_+ E^{-1} + \Psi_+ E(\delta S_+)E^{-1} = \Phi_+ E[\gamma(\infty,\tau) + \gamma(k)\mathsf{M}_{(33)}]E^{-1}\delta z \equiv \Phi_+ E\Pi E^{-1}\delta z, \quad (3.6)$$

where

$$\Pi(k) = \begin{pmatrix} \gamma_{11}(\infty,\tau) & \gamma_{12}(\infty,\tau) & \gamma_{13}(-\infty,\tau) \\ \gamma_{21}(\infty,\tau) & \gamma_{22}(\infty,\tau) & \gamma_{23}(-\infty,\tau) \\ \gamma_{31}(\infty,\tau) & \gamma_{32}(\infty,\tau) & \gamma_{33}(-\infty,\tau) \end{pmatrix}(k). \quad (3.7)$$

The matrix $\Pi(k)$ plays the key role in the following analysis because it contains all needed information about the perturbation. Its behavior with respect to the spectral parameter k reads

$$\Pi(k) = \Pi_r(k) + \frac{1}{k-k_1}\mathrm{Res}\{\Pi(k),k_1\}, \quad (3.8)$$

where $\Pi_r(k)$ is the holomorphic part in the upper half-plane $\mathrm{Im}k>0$ and Res{ , } stands for residue at k_1. At the asymptotics $\tau\to\infty$ the matrix Π is considerably simplified

$$\Pi(k) = \begin{pmatrix} 0 & 0 & \gamma_{13}(k) \\ 0 & 0 & \gamma_{23}(k) \\ 0 & 0 & \gamma_{33}(k) \end{pmatrix}, \quad \tau\to\infty. \quad (3.9)$$

Now we have everything to derive the equations determining z evolution of the discrete spectrum of the RH problem. As shown in the Appendix, they read

$$|p\rangle_z = ik_1^2 A|p\rangle - E(k_1)\Pi_r(k_1)E^{-1}(k_1)|p\rangle, \quad (3.10)$$

$$k_{1z} = -\mathrm{Res}\{\mathrm{tr}\Pi(k),k_1\} = -\mathrm{Res}\{\gamma_{33}(k),k_1\}. \quad (3.11)$$

In Sec. IV, on the example of combined linear and cubic perturbations, we give a detailed analysis of Eqs. (3.10) and (3.11).

IV. ADIABATIC APPROXIMATION

As pointed out in Sec. II, within the adiabatic approximation we should impose $\Phi_+ = \Gamma$, where [see Eq. (2.13)]

$$\Gamma = \mathbb{1} - \frac{k_1-k_1^*}{k-k_1^*}P, \quad \Gamma^{-1} = \mathbb{1} + \frac{k_1-k_1^*}{k-k_1}P. \quad (4.1)$$

This leads to the following expression for the matrix $\gamma(k)$ (3.3):

$$\gamma(k) = i\int_{-\infty}^{\infty} d\tau\, E^{-1}\left[R\left(\mathbb{1} - \frac{k_1-k_1^*}{k-k_1^*}P\right) + \frac{k_1-k_1^*}{k-k_1}PR\left(\mathbb{1} - \frac{k_1-k_1^*}{k-k_1^*}P\right)\right]E. \quad (4.2)$$

Hence,

$$\mathrm{Res}\{\gamma(k),k_1\} = i(k_1-k_1^*)\int_{-\infty}^{\infty} d\tau\, E^{-1}(k_1)PR(\mathbb{1}-P)E(k_1) \quad (4.3)$$

and the z dependence (3.11) of zero k_1 has the form

$$k_{1z} = 2\eta\left[\int_{-\infty}^{\infty} d\tau\, PR(\mathbb{1}-P)\right]_{33}. \quad (4.4)$$

Let us now explicitly obtain the value of the regular part $\gamma_r(k)$ at $k=k_1$, because just this value enters the matrix $\Pi_r(k_1)$ involved in Eq. (3.10). It follows from Eqs. (4.2) and (4.3):

$$\gamma_r(k_1)=[\gamma(k)-(k-k_1)^{-1}\mathrm{Res}\{\gamma(k),k_1\}]_{k_1}$$
$$=i\int_{-\infty}^{\infty}d\tau\, E^{-1}(k_1)PR(\mathbb{1}-P)E(k_1)$$
$$+i\lim_{k\to k_1}\frac{k_1-k_1^*}{k-k_1}\int_{-\infty}^{\infty}d\tau\left\{E^{-1}(k)PR\left(\mathbb{1}-\frac{k_1-k_1^*}{k-k_1^*}P\right)\right.$$
$$\left.\times E(k)-E^{-1}(k_1)PR(\mathbb{1}-P)E(k_1)\right\}$$
$$=i\int_{-\infty}^{\infty}d\tau\left\{E^{-1}(k_1)R(\mathbb{1}-P)E(k_1)+(k_1-k_1^*)\right.$$
$$\left.\times\frac{\partial}{\partial k}\left[E^{-1}(k)PR\left(\mathbb{1}-\frac{k_1-k_1^*}{k-k_1^*}P\right)E(k)\right]_{k_1}\right\}$$
$$=i\int_{-\infty}^{\infty}d\tau\, E^{-1}(k_1)\{R(\mathbb{1}-P)+PRP-i\tau(k_1-k_1^*)$$
$$\times[A,PR(\mathbb{1}-P)]\}E(k_1). \tag{4.5}$$

Taking into account Eq. (4.5) and rewriting by entries the formula (3.10),

$$p_{jz}=ik_1^2p_j-e^{2ik_1\tau}\gamma_{rj3}(k_1)p_3,$$
$$p_{3z}=-[ik_1^2+\gamma_{r33}(k_1)]p_3,$$

we obtain the corresponding equations for the quantities n_j, $j=1,2$, which describe the polarization state of the soliton

$$n_{jz}=[2ik_1^2+\gamma_{r33}(k_1)]^*n_j-[e^{2ik_1\tau}\gamma_{rj3}(k_1)]^*.$$

Hence it follows [compare with Eqs. (2.17) and (2.18)] that

$$n_{jz}^0=[\gamma_{r33}(k_1)]^*n_j^0-\exp\left[2i\int^z dzk_1^{*2}(z)\right][\gamma_{rj3}(k_1)]^*, \tag{4.6}$$

where the possible perturbation-induced z dependence of k_1 is taken into account.

Therefore, to describe the parameters of a perturbed soliton in the adiabatic approximation, it is necessary to solve Eqs. (4.4) and (4.6), provided the perturbation is a given function of τ and z. As an example, let us consider the perturbed Manakov system

$$\begin{aligned} &iq_{1z}+\tfrac{1}{2}q_{1\tau\tau}+(|q_1|^2+|q_2|^2)q_1=i\epsilon q_1-i\beta|q_1|^2q_1,\\ &iq_{2z}+\tfrac{1}{2}q_{2\tau\tau}+(|q_2|^2+|q_1|^2)q_2=i\epsilon q_2-i\beta|q_2|^2q_2,\end{aligned} \tag{4.7}$$

where ϵ and β are small real parameters. This means that the perturbation matrix R is given by

$$R=\epsilon Q-\beta Q^3. \tag{4.8}$$

Depending on the signs of ϵ and β this perturbation describes the action of linear and nonlinear damping or gain [34,35]. For example, for positive ϵ and β the perturbation (4.8) describes combined action of excess linear gain and nonlinear damping.

We start with the evolution of zero $k_1=\xi+i\eta$ of the RH problem. According to the formulas in Sec. II we have

$$P=\frac{1}{2}\begin{pmatrix}|\Theta_1|^2e^{-y} & \Theta_1^*\Theta_2e^{-y} & \Theta_1^*e^{i\phi}\\ \Theta_2^*\Theta_1e^{-y} & |\Theta_2|^2e^{-y} & \Theta_2^*e^{i\phi}\\ \Theta_1e^{-i\phi} & \Theta_2e^{-i\phi} & e^{y}\end{pmatrix}\mathrm{sech}y,\quad \epsilon Q=-2i\epsilon\eta\begin{pmatrix}0&0&\Theta_1^*e^{i\phi}\\ 0&0&\Theta_2^*e^{i\phi}\\ \Theta_1e^{-i\phi}&\Theta_2e^{-i\phi}&0\end{pmatrix}\mathrm{sech}y,$$

$$\beta Q^3=-8i\beta\eta^3\begin{pmatrix}0&0&-\Theta_1^*|\Theta_1|^2e^{i\phi}\\ 0&0&-\Theta_2^*|\Theta_2|^2e^{i\phi}\\ \Theta_1|\Theta_1|^2e^{-i\phi}&\Theta_2|\Theta_2|^2e^{-i\phi}&0\end{pmatrix}\mathrm{sech}^3y. \tag{4.9}$$

Hence, according to Eq. (4.4) we find

$$k_{1z}=2i\epsilon\eta-\frac{16}{3}i\beta\eta^3(|\Theta_1|^4+|\Theta_2|^4). \tag{4.10}$$

From Eq. (4.10) it follows that $\xi_z=0$, i.e., in the adiabatic approximation the velocity of the soliton is not affected by the perturbation (4.8), while the amplitude η satisfies the equation

$$\eta_z=2\epsilon\eta-\frac{16}{3}\beta\eta^3(|\Theta_1|^4+|\Theta_2|^4). \tag{4.11}$$

Now we turn to the other soliton parameters, i.e., Θ_1, Θ_2, and α. Since they are expressed through the quantities n_j^0, $j=1,2$, let us consider Eq. (4.6). From Eqs. (4.5) and (4.9) we get

$$\gamma_{r33}(k_1)=-\epsilon+\tfrac{8}{3}\beta\eta^2(|\Theta_1|^4+|\Theta_2|^4)$$

and

$$\exp\left(-2i\int^z dz k_1^2\right)\gamma_{rj3}(k_1)$$
$$=-\epsilon(n_j^0)^*\left[1+2\left(\alpha-4\xi\int^z dz\,\eta\right)\right]$$
$$+\frac{8}{3}\beta\eta^2(n_j^0)^*\left[|\Theta_j|^2+2\left(\alpha-4\xi\int^z dz\,\eta\right)\right.$$
$$\left.\times(|\Theta_1|^4+|\Theta_2|^4)\right],$$

where $\eta(z)$ is the solution of Eq. (4.11). Therefore, Eq. (4.6) takes the form

$$n_{jz}^0=2\left\{\epsilon\left(\alpha-4\xi\int^z dz\,\eta\right)-\frac{4}{3}\beta\eta^2\left[|\Theta_j|^2\right.\right.$$
$$\left.\left.+2\left(\alpha-4\xi\int^z dz\,\eta-\frac{1}{2}\right)(|\Theta_1|^4+|\Theta_2|^4)\right]\right\}n_j^0. \tag{4.12}$$

Then the evolution equation for α, due to Eq. (2.18), is written as

$$\alpha_z=2\left(\alpha-4\xi\int^z dz\,\eta\right)\left[\epsilon-\frac{8}{3}\beta\eta^2(|\Theta_1|^4+|\Theta_2|^4)\right] \tag{4.13}$$

and, in virtue of $\Theta_j=n_j^0 e^{-\alpha}$, the evolution equation for $|\Theta_1|$ is

$$|\Theta_1|_z=\tfrac{16}{3}\beta\eta^2|\Theta_1|(|\Theta_1|^2-1)(|\Theta_1|^2-\tfrac{1}{2}), \tag{4.14}$$

while $|\Theta_2|$ can be found from the identity $|\Theta_1|^2+|\Theta_2|^2=1$. As for the phase φ_j of Θ_j, $\Theta_j\equiv|\Theta_j|e^{i\varphi_j}$, we evidently have $\varphi_{jz}=0$ because the coefficients in the Eq. (4.12) are real valued. Thus, the evolution of parameters of the perturbed soliton within the adiabatic approximation is described by the closed system of Eqs. (4.11), (4.13), and (4.14). Standard analysis of the key equation (4.14) gives three stationary solutions for $|\Theta_1|$: $|\Theta_1|=0$, $|\Theta_1|=1/\sqrt{2}$, and $|\Theta_1|=1$. From Eq. (4.14) it is also seen that for $\beta>0$ the stationary point $|\Theta_1|=1/\sqrt{2}$ is the attractor for the initial values of $|\Theta_1|$ subjected to inequality $0<|\Theta_1|<1$, while for $\beta<0$ the stationary points $|\Theta_1|=0$ and $|\Theta_1|=1$ are the attractors for the initial values subjected to the inequalities $0\leqslant|\Theta_1|<1/\sqrt{2}$ and $1/\sqrt{2}<|\Theta_1|\leqslant 1$, respectively. Similar results were obtained by Afanasjev [35], who used an *Ansätz* to derive equations for the soliton parameters. It is essential that the rest equations (4.11) and (4.13) also have a stationary point for η provided $|\Theta_1|$ is equal to one of the stationary points discussed above. Indeed, denoting $\delta=|\Theta_1|^4+|\Theta_2|^4$ we get

$$\eta_z=2\,\eta(\epsilon-\tfrac{8}{3}\beta\delta\eta^2),$$
$$\alpha_z=2\left(\alpha-4\xi\int^z dz\,\eta\right)(\epsilon-\tfrac{8}{3}\beta\delta\eta^2), \tag{4.15}$$

and the stationary point η_* is given by

$$\eta_*^2=\frac{3\epsilon}{8\delta\beta},\qquad \delta=\begin{cases}1, & |\Theta_1|=0,1\\ \frac{1}{2}, & |\Theta_1|=\frac{1}{\sqrt{2}}\end{cases} \tag{4.16}$$

provided the condition $\epsilon\beta>0$ is satisfied. To consider the stability of the stationary solution η_*, turn to the first of Eqs. (4.15). For a small deviation $\Delta\eta$ from the stationary value we obtain the linearized equation

$$\Delta\eta_z=-4\epsilon\Delta\eta.$$

Hence, for positive ϵ (and, due to the condition $\epsilon\beta>0$, for positive β) the stationary point η_* is stable, otherwise it is unstable.

Thus, the analysis of the adiabatic approximation gives evidence of the existence of the stationary regimes of soliton propagation, with the perturbation of the form (4.8) determining the only amplitude (4.16) of the stationary soliton. This fact will be used in Sec. V for the calculation of corrections to the soliton shape.

V. FIRST-ORDER APPROXIMATION: DISTORTION OF THE SOLITON SHAPE

Within the adiabatic approximation we find the perturbation-induced evolution of the soliton parameters, but neglect a distortion of the soliton shape. Mathematically this means that we dealt before with discrete spectral data only. That is the continuous part of spectral data which describes a distortion of the soliton shape and possible emission of linear waves by soliton.

To take into account continuous spectral data we need to drop the condition $G=\mathbb{1}$ [and, as a consequence, $\Phi^0(k)=\mathbb{1}$] used in Sec. II. This necessitates the solution of the regular RH problem (2.12). Indeed, we can write in the first-order approximation

$$\Phi^0=\mathbb{1}+\Phi^{0(1)}, \tag{5.1}$$

where $\Phi^{0(1)}$ is a first-order correction in respect to perturbation. Then Eqs. (2.9) and (2.10) give

$$Q=-\lim_{k\to\infty}k[A,\Phi_+]=-\lim_{k\to\infty}k[A,(\mathbb{1}+\Phi_+^{0\,(1)})\Gamma]=Q_s+Q_s^{(1)},$$

where the correction $Q_s^{(1)}$ has the form $Q_s^{(1)}=-\lim_{k\to\infty}k[A,\Phi_+^{0\,(1)}\Gamma]$. Now the matrix G which determines the RH problem (2.12) also can be represented as $G=\mathbb{1}+G^{(1)}$, where $G^{(1)}$ is of the first order. Therefore, the regular RH problem in the first-order approximation takes the form

$$(\Phi_-^0)^{-1}\Phi_+^0=G^0=\mathbb{1}+\Gamma G^{(1)}\Gamma^{-1}. \tag{5.2}$$

The jump of the piecewise holomorphic function Φ^0 on the real axis $\mathrm{Im}k=0$ is expressed through the right-hand side of Eq. (5.2)

$$\Phi_+^0-\Phi_-^0=\Phi_-^0G^0-\Phi_-^0=\Phi_-^0\Gamma G^{(1)}\Gamma^{-1}.$$

The Plemelj formula gives

$$\Phi^0 = \mathbb{1} + \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{dl}{l-k}(\Phi_-^0\Gamma G^{(1)}\Gamma^{-1})(l). \qquad (5.3)$$

Selecting then in $\Phi_\pm^0$, due to Eq. (5.1), the contribution of the first-order approximation we obtain from Eq. (5.3)

$$\Phi_+^{0\,(1)} = \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{dl}{l-k}(\Gamma G^{(1)}\Gamma^{-1})(l).$$

Hence it follows that

$$Q_s^{(1)} = -\lim_{k\to\infty} k\left[A, \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{dl}{l-k}(\Gamma G^{(1)}\Gamma^{-1})(l)\Gamma(k)\right] \qquad (5.4a)$$

or, in view of the explicit expression for Γ (4.1),

$$q_j^{(1)} = \frac{i}{\pi}\int_{-\infty}^{\infty} dk[\Gamma G^{(1)}\Gamma^{-1}]_{3j}. \qquad (5.4b)$$

Thus, to obtain the correction to the soliton shape we have to calculate the matrix $G^{(1)}$.

The matrix $G^{(1)}$ can be calculated in the following way. Similar to Φ_+ [see Eq. (A3)] it is easy to show that Φ_-^{-1} involved in the formulation of the RH problem (2.8) satisfies the equation

$$\Phi_-^{-1}{}_z = ik^2 A\Phi_-^{-1} - \Phi_-^{-1}V - E\bar{\Pi}E^{-1}\Phi_-^{-1},$$

where $\bar{\Pi}(k) = -\Pi^\dagger(k^*)$. Hence, it follows that z evolution of G reads

$$G_z = ik^2[A,G] + GE\Pi E^{-1} - E\bar{\Pi}E^{-1}G. \qquad (5.5)$$

Now introduce the matrix $G_0(k) = \exp(-ikA\tau - ik^2Az)G\exp(ikA\tau + ik^2Az)$ and take into account that the continuous spectrum of the RH problem corresponds to the real axis $\mathrm{Im}k = 0$. Therewith $\bar{\Pi}(k) = -\Pi^\dagger(k)$ and

$$G_{0z} = G_0 e^{-ik^2Az}\Pi e^{ik^2Az} + e^{-ik^2Az}\Pi^\dagger e^{ik^2Az}G_0. \qquad (5.6)$$

Recalling that $\Pi(k)$ is of the first order in respect to perturbation and having in mind the representation $G_0 = \mathbb{1} + G_0^{(1)}$, we write Eq. (5.6) in the first order as

$$G_0^{(1)}{}_z = e^{-ik^2Az}(\Pi + \Pi^\dagger)e^{ik^2Az}. \qquad (5.7)$$

Here

$$\Pi + \Pi^\dagger = \begin{pmatrix} 0 & 0 & \gamma_{13} \\ 0 & 0 & \gamma_{23} \\ -\gamma_{31} & -\gamma_{32} & 0 \end{pmatrix} \qquad (5.8)$$

and we took into account that for $\mathrm{Im}k = 0$ in the first-order approximation $\gamma(k)$ satisfies the identity $\gamma_{\cdot 3}^*(k) = -\gamma_{3\cdot}(k)$. Integrating Eq. (5.7) for γ_{ij} determined in Eq. (4.2) and putting

$$G^{(1)}(k) = \exp(ikA\tau + ik^2Az)G_0^{(1)}\exp(-ikA\tau - ik^2Az), \qquad (5.9)$$

we can calculate the needed correction with the help of Eqs. (5.4). Note that Eq. (5.4a) can be represented in more detail provided the explicit expressions (4.1), (4.9), and Hermiticity of G are taken into account, namely,

$$\begin{aligned}(\Gamma G^{(1)}\Gamma^{-1})_{31} = {} & G_{31}^{(1)} + \frac{i\eta}{k-k_1}\Theta_1(\Theta_1^* G_{31}^{(1)} \\ & + \Theta_2^* G_{32}^{(1)})e^{-y}\mathrm{sech}y - \frac{i\eta}{k-k_1^*}G_{31}^{(1)}e^{y}\mathrm{sech}y \\ & + \frac{\eta^2}{(k-k_1)(k-k_1^*)}\Theta_1[\Theta_1^* G_{31}^{(1)} + \Theta_2^* G_{32}^{(1)} \\ & + (\Theta_1(G_{31}^{(1)})^* + \Theta_2(G_{32}^{(1)})^*)e^{-2i\phi}]\mathrm{sech}^2 y.\end{aligned} \qquad (5.10)$$

The corresponding expression for $(\Gamma G^{(1)}\Gamma^{-1})_{32}$ follows from Eq. (5.10) after the evident substitutions $\Theta_1 \leftrightarrow \Theta_2$ and $G_{31} \leftrightarrow G_{32}$. It follows from Eq. (5.10) that for $y \gg 1$ it is sufficient to consider the contribution of the first and third terms, whereas for $-y \gg 1$ the first and second ones contribute. Considering separately the linear and nonlinear parts of the perturbation (4.8) we find the corresponding values of $\gamma_{3j}(k)$, $\tilde{\gamma}_{3j}(k) \equiv \Theta_j^{-1}\exp[4ik\xi z - 2i(\xi^2+\eta^2)z]\gamma_{3j}(k)$

$$\tilde{\gamma}_{3j}^{(\epsilon)}(k) = -\epsilon\pi\exp\left(i\frac{k-\xi}{\eta}\alpha\right)\mathrm{sech}\left(\frac{\pi(k-\xi)}{2\eta}\right),$$

$$\begin{aligned}\tilde{\gamma}_{3j}^{(\beta)}(k) = {} & -\frac{2}{3}\pi\beta(k-k_1)(k-k_1^*)\left[2|\Theta_j|^2\left(1+\frac{i\eta}{k-k_1}\right)\right. \\ & \left. -(|\Theta_1|^4 + |\Theta_2|^4)\left(1+\frac{2i\eta}{k-k_1^*}\right)\right]\mathrm{sech}\left(\frac{\pi(k-\xi)}{2\eta}\right).\end{aligned}$$

Inserting these expressions in Eq. (5.7) and integrating on z we get $(G_0^{(1)})_{3j}$ for both perturbations. Thereby, omitting for $z \gg 1$ the fast oscillating exponent, we obtain from Eq. (5.9)

$$\begin{aligned}G_{3j}^{(1)}(\epsilon) = {} & -\frac{i\epsilon\pi\Theta_j}{2(k-k_1)(k-k_1^*)}\exp\left[2i(\xi^2+\eta^2)z - i\frac{\xi\alpha}{\eta}\right] \\ & \times\exp\left(-\frac{ik}{\eta}y\right)\mathrm{sech}\left(\frac{\pi(k-\xi)}{2\eta}\right),\end{aligned} \qquad (5.11a)$$

$$\begin{aligned}G_{3j}^{(1)}(\beta) = {} & -\frac{i\beta\pi\Theta_j}{3(k-k_1)(k-k_1^*)}\exp\left[2i(\xi^2+\eta^2)z - i\frac{\xi\alpha}{\eta}\right] \\ & \times\exp\left(-\frac{ik}{\eta}y\right)\left[2|\Theta_j|^2\left(1+\frac{i\eta}{k-k_1}\right)\right. \\ & \left. -(|\Theta_1|^4 + |\Theta_2|^4)\left(1+\frac{2i\eta}{k-k_1^*}\right)\right]\mathrm{sech}\left(\frac{\pi(k-\xi)}{2\eta}\right),\end{aligned} \qquad (5.11b)$$

where in the case of the nonlinear perturbation, i.e., in Eq. (5.11b), we ought to take for $|\Theta_j|$ their asymptotic values in accordance with the results of the adiabatic approximation in Sec. IV. Now insert the obtained expressions (5.11) into Eq. (5.10) and integrate on k according to Eq. (5.4b). All the integrals are easily calculated by means of residues, the contributions from the poles $\xi\pm 3i\eta$, $\xi\pm 5i\eta$, etc., of $\operatorname{sech}[\pi(k-\xi)/2\eta]$ being negligibly small as compared with the contributions of the poles $\xi\pm i\eta$. Here, after simple calculations we get the corrections to the soliton shape for $y\gg 1$

$$q_j^{(1)}(\epsilon)=-\frac{\epsilon}{2\eta}\Theta_j \exp\left[2i(\xi^2+\eta^2)z-\frac{i\xi}{\eta}(y+\alpha)\right]\times\left(2y^2+\frac{\pi^2}{6}\right)e^{-y}, \quad (5.12a)$$

$$q_j^{(1)}(\beta)=\frac{4}{3}\beta\eta\Theta_j \exp\left[2i(\xi^2+\eta^2)z-\frac{i\xi}{\eta}(y+\alpha)\right]\times\left[\left(2y^2+\frac{\pi^2}{6}-1\right)(|\Theta_1|^4+|\Theta_2|^4)+2(1-y)|\Theta_1|^2\right]e^{-y}. \quad (5.12b)$$

Note that the correction (5.12a) for $|\Theta_j|=1$, what corresponds to the NLS equation, coincides with the result obtained in Ref. [30]. Now let us turn to the case of the joint action of the two perturbations. Under the condition $\epsilon\beta>0$, there exists the stationary regime of the soliton propagation with the amplitude satisfying Eq. (4.15). For example, for positive ϵ and β we simply ought to sum the two corrections (5.12) with $|\Theta_1|=|\Theta_2|=1/\sqrt{2}$ and $\eta=\eta_*$ from Eq. (4.15). The resulting steady-state correction to the soliton shape reads for $y\gg 1$

$$q_j^{(1)}=\frac{\epsilon}{2\sqrt{2}\eta}\exp\left[2i(\xi^2+\eta^2)z-\frac{i\xi}{\eta}(y+\alpha)\right](1-2y)e^{-y}. \quad (5.13a)$$

Analogous simple calculations can be carried out for $-y\gg 1$. We obtain the following resulting steady-state correction to the soliton shape for $-y\gg 1$:

$$q_j^{(1)}=\frac{\epsilon}{2\sqrt{2}\eta}\exp\left[2i(\xi^2+\eta^2)z+\frac{i\xi}{\eta}(y-\alpha)\right](1+2y)e^{y}. \quad (5.13b)$$

These expressions demonstrate that in the process of going to the stationary regime (for ϵ and β positive) the two corrections to the soliton shape partly compensate each other, which results in mutual canceling out of the leading terms with y^2 of the asymptotics (5.12).

For negative ϵ and β the condition $\epsilon\beta>0$ also holds and the stationary solutions are $|\Theta_1|=1$ ($|\Theta_2|=0$) and $|\Theta_1|=0$ ($|\Theta_2|=1$), which lead to doubling of the corrections (5.13). But in this case the stationary point η_* is unstable.

VI. CONCLUSION

We have proposed a natural generalization of the Kaup-Karpman-Maslov perturbation scheme which was elaborated for soliton equations with 2×2 matrix spectral problem, to the Manakov system. It is important to stress that it is the Riemann-Hilbert ideology initiated many years ago for studying the soliton equations by Zakharov, Shabat, Manakov, and some others that turns out to be the best ground to build an analytical approach for dealing with small perturbations in soliton systems. It is evident that neither the Manakov system nor the Zakharov-Shabat spectral problem exhaust the applicability of the presented formalism. Recently it was proven [36] that the Riemann-Hilbert problem can be successfully used to treat perturbed solitons of equations associated with the $N\times N$ matrix Zakharov-Shabat spectral problem. Moreover, we developed [37,38] a Riemann-Hilbert-based method to find soliton solutions for the modified Manakov system (a system of two coupled modified nonlinear Shrödinger equations) solvable by the 3×3 matrix Wadati-Konno-Ichikawa spectral problem. The soliton perturbation theory for the modified Manakov system will be present elsewhere.

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APPENDIX

Here we derive the evolution equations (3.10) and (3.11) for discrete spectral data. For simplicity consider the case of a single zero k_1 of the RH problem. We start with the equation $\Phi_+(k_1)|p\rangle=0$ which is true irrespectively of the action of a perturbation. Differentiation of this equation gives

$$\Phi_+(k_1)\frac{d}{dz}|p\rangle+\left(\frac{d}{dz}\Phi_+(k)\right)_{k_1}|p\rangle=0. \quad (A1)$$

Note that z evolution of $\Phi_+(k)$, in the case of action of a perturbation, contains the additional term (3.6), i.e.,

$$\Phi_{+z}=V\Phi_+-ik^2\Phi_+A+\Phi_+\hat{\Pi}, \quad (A2)$$

where $\hat{\Pi}=E\Pi E^{-1}$ and the matrix Π is determined in Eq. (3.7). This gives

$$\frac{d}{dz}\Phi_+(k)=[\Phi_+(k)]_z+k_z\frac{\partial}{\partial k}\Phi_+(k)=V\Phi_+-ik^2\Phi_+A+\Phi_+\hat{\Pi}+k_z\frac{\partial}{\partial k}\Phi_+. \quad (A3)$$

In view of the explicit structure (3.8) of Π, we introduce a new matrix $\tilde{\Pi}$

$$\tilde{\Pi}(k)\equiv(k-k_1)\hat{\Pi}_r(k)+\operatorname{Res}\{\hat{\Pi}(k),k_1\}, \quad (A4)$$

hence

$$\tilde{\Pi}(k_1)|p\rangle=\operatorname{Res}\{\hat{\Pi}(k),k_1\}|p\rangle.$$

Equation (A2) gives

$$\hat{\Pi}(k)=\Phi_+^{-1}\Phi_{+z}+ik^2A-\Phi_+^{-1}V\Phi_+,$$

and, $\tilde{\Phi}_+\equiv(k-k_1)\Phi_+^{-1}$,

$$\tilde{\Pi}(k)=\tilde{\Phi}_+\Phi_{+z}+ik^2(k-k_1)A-\tilde{\Phi}_+V\Phi_+.$$

By virtue of $\Phi_+(k_1)|p\rangle=0$ and Eq. (A4) the above formula gives

$$\tilde{\Pi}(k_1)|p\rangle=\mathrm{Res}\{\hat{\Pi}(k),k_1\}|p\rangle=\{\tilde{\Phi}_+(k)[\Phi_+(k)]_z\}_{k_1}|p\rangle$$
$$=(k-k_1)_{z,k=k_1}|p\rangle=-k_{1z}|p\rangle.$$

Therefore, we obtain the important identity

$$\mathrm{Res}\{\hat{\Pi}(k),k_1\}|p\rangle=-k_{1z}|p\rangle, \tag{A5}$$

which will be used below. Equation (A1), with account of Eq. (A3), takes the form

$$\Phi_+(k_1)(|p\rangle_z-ik_1^2A|p\rangle)+(\Phi_+(k)[\hat{\Pi}_r(k)+(k-k_1)^{-1}$$
$$\times\mathrm{Res}\{\hat{\Pi}(k),k_1\}])_{k_1}|p\rangle+k_{1z}\left(\frac{\partial}{\partial k}\Phi_+(k)\right)_{k_1}|p\rangle=0. \tag{A6}$$

As $\Phi_+(k_1)\mathrm{Res}\{\hat{\Pi}(k),k_1\}=0$ in view of the existence of the quantity $(\Phi_+\hat{\Pi})_{k_1}$, we obtain

$$[\Phi_+(k-k_1)^{-1}\mathrm{Res}\{\hat{\Pi}(k),k_1\}]_{k_1}|p\rangle$$
$$=\left(\frac{\Phi_+(k)-\Phi_+(k_1)}{k-k_1}\right)_{k_1}\mathrm{Res}\{\hat{\Pi}(k),k_1\}|p\rangle$$
$$=\left(\frac{\partial}{\partial k}\Phi_+(k)\right)_{k_1}\mathrm{Res}\{\hat{\Pi}(k),k_1\}|p\rangle$$
$$=-k_{1z}\left(\frac{\partial}{\partial k}\Phi_+(k)\right)_{k_1}|p\rangle,$$

where during the last stage we used Eq. (A5). Now it is seen that this term is canceled with the last term in Eq. (A6), which gives

$$\Phi_+(k_1)(|p\rangle_z-ik_1^2A|p\rangle+\hat{\Pi}_r(k_1)|p\rangle)=0.$$

Because of the identities $\dim[\ker\Phi_+(k_1)]=1$ and $\Phi_+(k_1)\mathrm{Res}\{\hat{\Pi}(k),k_1\}=0$, we obtain the perturbation-induced z-evolution equation for the vector $|p\rangle$

$$|p\rangle_z=ik_1^2A|p\rangle-E(k_1)\Pi_rE^{-1}(k_1)|p\rangle, \tag{A7}$$

where as $\Pi(k)$ we may take its asymptotic expression (3.9).

Now turn to the z evolution of zero k_1. As $\det\Phi_+(k)=O(k-k_1)$ for $k\to k_1$, it follows from the evident expression $d/dz[\det\Phi_+(k)]_{k_1}=0$ that

$$k_{1z}=-\left[\frac{[\det\Phi_+(k)]_z}{\frac{\partial}{\partial k}\det\Phi_+(k)}\right]_{k_1}.$$

Taking into account $[\det\Phi_+(k)]_z=[\mathrm{tr}\Pi(k)]\det\Phi_+(k)$ and

$$\det\Phi_+(k)=(k-k_1)\det\Phi_+^0(k),\quad \det\Phi_+^0(k)\neq0,$$

we get

$$k_{1z}=-\lim_{k\to k_1}\frac{\mathrm{tr}[\Pi_r(k)+(k-k_1^*)^{-1}\mathrm{Res}\{\hat{\Pi}(k),k_1\}]\det\Phi_+^0(k)}{\left[\frac{\partial}{\partial k}\det\Phi_+^0(k)\right]\frac{k-k_1}{k-k_1^*}+\det\Phi_+^0(k)\frac{\partial}{\partial k}\frac{k-k_1}{k-k_1^*}}$$
$$=-\mathrm{Res}\{\mathrm{tr}\hat{\Pi}(k),k_1\}=-\mathrm{Res}\{\gamma_{33}(k),k_1\},$$

i.e., Eq. (3.11).

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